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THE INTERACTION OF AN ACOUSTIC WAVE AND AN ELASTIC SPHERICAL SHELL

by

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The Interaction of an Acoustic Wave and an Elastic Spherical Shell[1]

By

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I. Introduction.

The problem to be considered here, that of a plane pressure wave impinging on a thin spherical shell, was suggested by G. F. Carrier in consequence of work previously done by him on a related problem [1]. (cf. also [6].)

An attempt had been made to determine the response to an incident acoustic wave of a thin elastic shell, in particular a cylindrical shell, taking into account both the incident and diffracted waves. The form of the functions dealt with in the analysis made it difficult to obtain accurate explicit results. If the obstacle is taken to be spherical in shape, we still have a fairly practical though highly simplified model of an actual physical structure; moreover, the problem becomes mathematically simpler, admitting of exact solutions for the deformation and accompanying strains in the elastic body. It was therefore decided to treat the case of the sphere in detail.

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2. Research Associate, Brown University.

We will, as for the cylinder, deal with the linearized theory of wave propagation in a compressible fluid, and with small deflections of the shell.

II. Forced Vibrations of A Thin Spherical Shell.

We consider a closed shell of thickness h with $h<<R$ where R is the radius of the middle surface. The motion of any closed oval[1] shell, in particular a spherical shell is, by a theorem of Jellett [2], primarily extensional. Therefore the general membrane theory of shells is applicable. If we locate the origin of our coordinate system at the center of the shell, and choose as the z-axis the direction of propagation of the incoming wave, then we have the additional simplification of symmetrical loading (c.f. Fig. 1). The equations of dynamic equilibrium for an element of shell may therefore be written [3]

$$\frac{\partial}{\partial\theta}(N_\theta R \sin\theta) - N_\varphi R \cos\theta - \rho h \frac{\partial^2 v}{\partial t^2} R^2 \sin\theta = 0 \qquad (2.1)$$

$$N_\theta + N_\varphi = (s - \rho h \frac{\partial^2 w}{\partial t^2})R = 0. \qquad (2.2)$$

Here N_φ and N_θ are the normal forces/unit length acting on the sides of the element, s is the applied force (radial in direction), ρ is the shell density, and v and w are the tangential and radial components of the displacement.

To eliminate N_φ and N_θ from equations (2.1) and (2.2) we make use of Hooke's Law

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1. Principle radii of curvature finite and of the same sign.

$$\varepsilon_{\theta\theta} = \frac{1}{Eh}(N_\theta - \nu N_\varphi) \tag{2.3}$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{Eh}(N_\varphi - \nu N_\theta) \tag{2.4}$$

and the expressions for the strains in spherical coordinates

$$\varepsilon_{\theta\theta} = \frac{1}{R}\left[\frac{\partial v}{\partial \theta} + w\right] \tag{2.5}$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{R}\left[v \cot\theta + w\right]. \tag{2.6}$$

Combining (2.3), (2.4), (2.5) and (2.6), we obtain N_φ and N_θ in terms of the displacements w and v

$$N_\varphi + N_\theta = \frac{Eh}{1-\nu}\left[\frac{v}{R}\cot\theta + 2\frac{w}{R} + \frac{1}{R}\frac{\partial v}{\partial \theta}\right] \tag{2.7}$$

$$N_\theta - N_\varphi = -\frac{Eh}{1+\nu}\left[\frac{v}{R}\cot\theta - \frac{1}{R}\frac{\partial v}{\partial \theta}\right] \tag{2.8}$$

so that the equations of equilibrium become

$$\frac{Eh}{1-\nu^2}\left[\nu\left(-v + \frac{\partial w}{\partial \theta}\right) + \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial \theta}\operatorname{ctn}\theta - v\operatorname{ctn}^2\theta\right] - \rho h \frac{\partial^2 v}{\partial t^2} R^2 = 0 \tag{2.9}$$

$$\frac{Eh}{1-\nu}\left[v\operatorname{ctn}\theta + 2w + \frac{\partial v}{\partial \theta}\right] - \left[s - \rho h \frac{\partial^2 w}{\partial t^2}\right]R^2 = 0. \tag{2.10}$$

An equation in v only can be easily obtained as follows. Consider the operator

$$M_o = 2 + \frac{(1-\nu)R^2}{E}\frac{\partial^2}{\partial t^2}. \tag{2.11}$$

$M_o(w)$ is found from (2.10) to be equal to

$$-v\operatorname{ctn}\theta - \frac{\partial v}{\partial \theta} + \frac{1-\nu}{Eh} R^2 s. \tag{2.12}$$

If M_o is then applied to (2.9), and (2.12) substituted wherever $M_o(w)$ appears, a relation in v only results:

$$M_o(2.9) = \frac{2(1-\nu^2)\rho R^2}{E}\frac{\partial^2 v}{\partial t^2} + \frac{(1-\nu^2)(1-\nu)\rho^2 R^4}{E^2}\frac{\partial^4 v}{\partial t^4} + 2\nu v + \frac{\nu(1-\nu)\rho R^2}{E}\frac{\partial^2 v}{\partial t^2} - 2\frac{\partial^2 v}{\partial \theta^2} - \frac{(1-\nu)\rho R^2}{E}\frac{\partial^4 v}{\partial t^2 \partial \theta^2} + 2v\,\mathrm{ctn}^2\theta + \frac{(1-\nu)\rho R^2}{E}\mathrm{ctn}^2\theta\,\frac{\partial^2 v}{\partial t^2} - 2\,\mathrm{ctn}\,\theta\,\frac{\partial v}{\partial \theta} - \frac{(1-\nu)\rho R^2}{E}\frac{\partial^3 v}{\partial \theta\,\partial t^2}\mathrm{ctn}\,\theta - [1+\nu][-\mathrm{ctn}\,\theta\,\frac{\partial v}{\partial \theta} + v\,\csc^2\theta - \frac{\partial^2 v}{\partial \theta^2} + \frac{(1-\nu)R^2}{Eh}\frac{\partial s}{\partial \theta}] = 0. \qquad (2.13)$$

A similar procedure[2] is used to arrive at an equation for w:

$$2w - \frac{1-\nu}{Eh}R^2(s - \rho h\frac{\partial^2 w}{\partial t^2}) - \frac{2(1+\nu)\rho R^2}{E}\frac{\partial^2 w}{\partial t^2} + \frac{(1-\nu^2)\rho R^4}{E^2 h}\frac{\partial^2 s}{\partial t^2} - \frac{(1-\nu^2)\rho^2 R^4}{E^2}\frac{\partial^4 w}{\partial t^4} + \mathrm{ctn}\,\theta\,\frac{\partial w}{\partial \theta} + \frac{\partial^2 w}{\partial \theta^2} - \frac{R^2}{Eh}\mathrm{ctn}\,\theta\,\frac{\partial s}{\partial \theta} - \frac{R^2}{Eh}\frac{\partial^2 s}{\partial \theta^2} + \frac{R^2\rho}{E}\mathrm{ctn}\,\theta\,\frac{\partial^3 w}{\partial t^2 \partial \theta} + \frac{\rho R^2}{E}\frac{\partial^4 w}{\partial t^2 \partial \theta^2} = 0. \qquad (2.14)$$

If we introduce the new variables

$$w^* = \frac{w}{R}; \quad v^* = \frac{v}{R}; \quad \tau^2 = \frac{t^2}{t_o^2} = \frac{Et^2}{R^2\rho}; \quad s^* = \frac{s}{s_o} = \frac{Rs}{Eh}$$

then (13) and (14) reduce to the simpler non-dimensional forms

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2. The operation $L_1(2.9) + L_2(2.10)$, where $L_1 = -(\mathrm{ctn}\,\theta + \frac{\partial}{\partial \theta})$ and $L_2 = 1 + \mathrm{ctn}\,\theta\,\frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} - \frac{(1+\nu)\rho R^2}{E}\frac{\partial^2}{\partial t^2}$, gives the desired result.

$$M(v^*) = 2(1-\nu^2)\frac{\partial^2 v^*}{\partial\tau^2} + (1-\nu^2)(1-\nu)\frac{\partial^4 v^*}{\partial\tau^4} + 2\nu v^* + \nu(1-\nu)\frac{\partial^2 v^*}{\partial\tau^2}$$
$$- 2\frac{\partial^2 v^*}{\partial\theta^2} - (1-\nu)\frac{\partial^4 v^*}{\partial\tau^2\partial\theta^2} + 2v^* \operatorname{ctn}^2\theta + +(1-\nu)\operatorname{ctn}^2\theta\frac{\partial^2 v^*}{\partial\tau^2}$$
$$- 2\operatorname{ctn}\theta\frac{\partial v^*}{\partial\theta} - (1-\nu)\operatorname{ctn}\theta\frac{\partial^3 v^*}{\partial\theta\partial\tau^2} + [1+\nu]\Big[\operatorname{ctn}\theta\frac{\partial v^*}{\partial\theta}$$
$$- v^*\csc^2\theta + \frac{\partial^2 v^*}{\partial\theta^2}\Big] = (1-\nu^2)\frac{\partial s^*}{\partial\theta} \qquad (2.15)$$

and

$$L(w^*) = \operatorname{ctn}\theta\cdot(\frac{\partial w^*}{\partial\theta} + \frac{\partial^3 w^*}{\partial\tau^2\partial\theta}) + 2w^* + \frac{\partial^2 w^*}{\partial\theta^2} + \frac{\partial^4 w^*}{\partial\tau^2\partial\theta^2}$$
$$- (1-\nu^2)\frac{\partial^4 w^*}{\partial\tau^4} + (1-\nu)\frac{\partial^2 w^*}{\partial\tau^2} - 2(1+\nu)\frac{\partial^2 w^*}{\partial\tau^2}$$
$$= \operatorname{ctn}\theta\frac{\partial s^*}{\partial\theta} - (1-\nu^2)\frac{\partial^2 s^*}{\partial\tau^2} + \frac{\partial^2 s^*}{\partial\theta^2} + (1-\nu)s^*. \qquad (2.16)$$

III. Acoustic Wave Propagation.

As in the cylindrical case [1] we have the acoustic wave equation

$$\Delta\varphi - \lambda^2\varphi_{\tau\tau} = 0 \qquad (3.1)$$

where φ is the velocity potential, $\Delta\varphi$ is the Laplacian in spherical coordinates and $\lambda^2 = \frac{R^2}{t_o^2 c^2} = \frac{E}{\rho c^2}$. ($c^2$ is the acoustic speed of the fluid.) The pressure perturbation p is again given by

$$p = -p^*\varphi_{,\tau}(r,\theta,\tau)$$
$$p^* = \frac{\rho_f R^2}{t_o^2} = \frac{\rho_f E}{\rho} \qquad (3.2)$$

where ρ_f is the fluid density. The applied stress s of (2.2) must of course be the same as the acoustic pressure p at the surface of the sphere. We have in fact

$$s^* = -\frac{p}{s_o} = \frac{p^*}{s_o}\varphi_{,\tau}(1,\theta,\tau) = \frac{\rho_f R}{\rho h}\varphi_{,\tau}(1,\theta,\tau) = \beta\varphi_{,\tau}(1,\theta,\tau).$$

IV. The Interaction Problem.

We now pose the following problem. An incoming plane wave, φ_o, obeying (3.1) impinges on an elastic spherical shell causing it to vibrate according to (2.15) and (2.16). The vibrating body acts as a scatterer and, to a lesser extent, as a radiator. The outgoing waves (scattered and radiated) also obeying (3.1), in turn influence the nature of the vibrations. At the surface of the shell, the radial velocity of the fluid, $\varphi_r(1,\theta,\tau)$, must be equal to the radial velocity of the shell, $w^*_\tau(\theta,\tau)$. We wish to determine the motion of the sphere and the pressure distribution associated with the incoming and outgoing waves.

The pressure associated with the incident wave is taken to be [1]

$$p = \begin{cases} Q_o e^{\delta(z-\tau/\lambda)} & z \le \tau/\lambda \\ 0 & z > \tau/\lambda \end{cases} \tag{4.1}$$

so that the initial velocity potential is

$$\varphi_o = \begin{cases} \frac{Q_o\lambda}{p^*\delta} e^{\delta(z-\tau/\lambda)} - \frac{Q_o\lambda}{p^*\delta} & z \le \tau/\lambda \\ 0 & z > \tau/\lambda. \end{cases} \tag{4.2}$$

For simplicity let

$$\varphi_o = \frac{Q_o}{p^*}\psi^o, \qquad \varphi = \frac{Q_o}{p^*}(\psi^o + \psi) = \frac{Q_o}{p^*}\chi$$
$$w^* = \frac{Q_o}{p^*}W, \qquad v^* = \frac{Q_o}{p^*}V. \tag{4.3}$$

Then our boundary value problem is defined by the following set of equations:[3]

$$L(W) = \beta[\chi_\tau'(1,\theta,\tau)\operatorname{ctn}\theta - (1-\nu^2)\chi_{\tau\tau\tau}(1,\theta,\tau) + \chi_\tau''(1,\theta,\tau) + (1-\nu)\chi_\tau(1,\theta,\tau)] \tag{4.4}$$

$$M(V) = \beta(1-\nu^2)\chi_\tau'(1,\theta,\tau) \tag{4.5}$$

$$\Delta\psi - \lambda^2\psi_{\tau\tau} = 0 \tag{4.6}$$

$$W_\tau(\theta,\tau) = \chi_r(1,\theta,\tau) \qquad \text{[Boundary Condition]} \tag{4.7}$$

$$\psi_o = \begin{cases} \frac{\lambda}{\delta}\, e^{\delta(z-\tau/\lambda)} - \frac{\lambda}{\delta} & z \le \tau/\lambda \\ 0 & z > \tau/\lambda\ . \end{cases} \tag{4.8}$$

These equations will be more easily handled if Fourier transforms are first introduced to eliminate the time dependence. Denote the transform of a function F by $\overline{F}$. Then F and $\overline{F}$ are related by:

$$\overline{F}(r,\theta,\eta-i\alpha) = \int_{-\infty}^{+\infty} F(r,\theta,\tau)e^{-\alpha\tau}e^{-i\eta\tau}\,d\tau \tag{4.9a}$$

where α is any positive real number.

$$F(r,\theta,\tau) = \frac{1}{2\pi}\, e^{\alpha\tau} \int_{-\infty}^{+\infty} \overline{F}(r,\theta,\eta-i\alpha)e^{i\eta\tau}\,d\eta$$

or, letting $\eta - i\alpha = \xi$

$$F(r,\theta,\tau) = \frac{1}{2\pi}\int_{-i\alpha-\infty}^{-i\alpha+\infty} F(r,\theta,\xi)e^{i\xi\tau}\,d\xi. \tag{4.9b}$$

This generalized definition of the Fourier transform has been used because the usual definition, $F(r,\theta,\eta) = \int_{-\infty}^{+\infty} F(r,\theta,\tau)e^{-i\eta\tau}\,d\tau$,

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3. The operators L and M which appear here are those given in (2.15) and (2.16).

fails in the case of the function χ.[4]

Applying (4.9a) to (3.4) through (3.8) we obtain

$$\bar{L}(\bar{W}) = i\beta\xi\,[\,\bar{\chi}'\ \mathrm{ctn}\ \theta + \bar{\chi}''(1,\theta,\xi) + (1-\nu)\bar{\chi} + \xi^2(1-\nu^2)\bar{\chi}\,] \tag{4.10}$$

where

$$\bar{L} = \Big\{\mathrm{ctn}\ \theta\ \Big(\frac{\partial}{\partial\theta} - \xi^2\,\frac{\partial}{\partial\theta}\Big) + 2 + \frac{\partial^2}{\partial\theta^2} - \xi^2\,\frac{\partial^2}{\partial\theta^2} - (1-\nu^2)\xi^4 - (1-\nu)\xi^2 + 2(1+\nu)\xi^2\Big\}$$

$$\bar{M}(\bar{V}) = i\beta\xi\ (1-\nu^2)\bar{\chi}' \tag{4.11}$$

where

$$\bar{M} = \Big\{-2(1-\nu^2)\xi^2 + (1-\nu^2)(1-\nu)\xi^4 + 2\nu - \nu(1-\nu)\xi^2 - 2\,\frac{\partial^2}{\partial\theta^2} + (1-\nu)\xi^2 + 2\ \mathrm{ctn}^2\ \theta - (1-\nu)\xi^2\ \mathrm{ctn}^2\theta - 2\ \mathrm{ctn}\ \theta\ \frac{\partial}{\partial\theta} + (1-\nu)\xi^2\ \mathrm{ctn}\ \theta\ \frac{\partial}{\partial\theta} + (1+\nu)\Big(\mathrm{ctn}\ \theta\ \frac{\partial}{\partial\theta} - \csc^2\theta + \frac{\partial^2}{\partial\theta^2}\Big)\Big\}$$

$$\Delta\,\bar{\psi} + \lambda^2\xi^2\bar{\psi} = 0 \tag{4.12}$$

$$i\xi\bar{W}(\theta,\xi) = \bar{\chi}_r(1,\theta,\xi) \tag{4.13}$$

$$\bar{\psi}^o = \frac{-e^{-i\lambda z\,\xi}}{i\xi(\delta/\lambda + i\xi)} = -f(\xi)e^{-i\lambda r\xi\cos\theta} \tag{4.14}$$

It can be shown, using the method of separation of variables on (4.12), that $\bar{\psi}$ is of the form

$$r^{-1/2}\,[\,J_{n+\frac{1}{2}}(\lambda\xi r) + CY_{n+\frac{1}{2}}(\lambda\xi r)\ P_n(\cos\theta)].$$

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4. It should be noted that for the functions we are interested in, the tendancy of the integrand in (4.9a) to become infinite for large negative τ is only apparent. Actually, $\psi^o = 0$ for $-\infty < \tau/\lambda < z$; and W, V, and ψ are zero until the wave hits the shell, i.e. for $-\infty < \tau/\lambda < -1$, so that integral of (4.9a) always exists.

Since the waves associated with $\bar{\psi}$ must be outwardly moving, take $C = -i$ and write:

$$\bar{\psi} = \sum_{n=0}^{\infty} A_n(\xi) h_n^{(2)}(\lambda\xi r) P_n(\cos\theta) \tag{4.15}$$

where

$$h_n^{(2)}(\lambda\xi r) = \left(\frac{\pi}{\lambda\xi r}\right)^{1/2} \left[J_{n+\frac{1}{2}}(\lambda\xi r) - iY_{n+\frac{1}{2}}(\lambda\xi r) \right].$$

$\bar{\psi}_o$ may be expanded similarly:

$$\bar{\psi}^o(r,\theta,\xi) = -f(\xi) \sum_{n=0}^{\infty} (2n+1)(-i)^n j_n(\lambda\xi r) P_n(\cos\theta) \tag{4.16}$$

$$= \sum_{n=0}^{\infty} B_n(\xi) j_n(\lambda\xi r) P_n(\cos\theta)$$

where

$$j_n(\lambda\xi r) = \left(\frac{\pi}{\lambda\xi r}\right)^{1/2} J_{n+\frac{1}{2}}(\lambda\xi r).$$

From the work of Lamb [4] we know that the complete solutions for $\bar{W}$ and $\bar{V}$ may be written:

$$\bar{W}(\theta,\xi) = \sum_{m=0}^{\infty} \bar{W}_n(\xi) P_n(\cos\theta) \tag{4.17}$$

$$\bar{V}(\theta,\xi) = -\sum_{n=0}^{\infty} \bar{V}_n(\xi) P_n^1(\cos\theta) = -\sum_{n=0}^{\infty} \bar{W}_n P_n'(\cos\theta)\sin\theta. \tag{4.18}$$

Substituting these expressions into (4.10), (4.11) and (4.13), we get three algebraic equations for the three unknowns $\bar{W}_n$, $\bar{V}_n$ and A_n.

$$[(1-\xi^2)(n^2+n) - 2 + (1-\nu^2)\xi^4 + (1-\nu)\xi^2 - 2(1+\nu)\xi^2]\bar{W}_n$$

$$= -i\beta\xi[(1-\nu) + (1-\nu^2)\xi^2 - (n^2+n)][A_n h_n^{(2)}(\lambda\xi) + B_n j_n(\lambda\xi)] \tag{4.19}$$

$$[2(1+\nu)\xi^2 - (1-\nu^2)\xi^4 + \nu\xi^2 + 1 + (n+n^2-1)(\xi^2-1)]\bar{V}_n$$

$$= -i\beta\xi(1+\nu)[A_n h_n^{(2)}(\lambda\xi) + B_n j_n(\lambda\xi)] \tag{4.20}$$

$$\lambda\xi B_n(\xi) j_n'(\lambda\xi) + \lambda\xi A_n(\xi) h_n^{(2)'}(\lambda\xi) = i\xi W_n \qquad (4.21)$$

These are readily solved to give:

$$W_n = \frac{(2n+1)(-i)^n C_2}{\lambda\xi(i\xi)^2(\frac{\delta}{\lambda} + i\xi)[C_1\lambda h_n^{(2)'} + iC_2 h_n^{(2)}]} \qquad (4.22)$$

$$V_n = -\frac{(2n+1)(-i)^n \beta(1+\nu)}{\lambda\xi(i\xi)(\frac{\delta}{\lambda} + i\xi)[C_1\lambda h_n^{(2)'} + iC_2 h_n^{(2)}]} \qquad (4.23)$$

$$A_n = -\frac{1}{2}\left[1 + \frac{C_1\lambda h_n^{(1)'} + iC_1 h_n^{(1)}}{C_1\lambda h_n^{(2)'} + iC_2 h_n^{(2)}}\right]\frac{1}{\xi^2}(2n+1)(-i)^n \qquad (4.24)$$

where

$$C_1 = (1-\xi^2)(n^2+n) - 2 + (1-\nu^2)\xi^4 + (1-\nu)\xi^2 - 2(1+\nu)\xi^2$$

$$C_2 = i\beta\xi[(1-\nu) + (1-\nu^2)\xi^2 - (n^2+n)].$$

The quantities of physical interest to us are the stresses and radial acceleration associated with each vibrational mode, and the total pressure distribution. These can all be found at least in principle from (4.22), (4.23), (4.24), (4.15) and the inversion formula (4.9b).

V. Numerical results for the shell.

The transforms of the stress components will all be linear combinations of $\bar{W}_n$ and $\bar{V}_n$ (in general, $g_1(\theta)\bar{W}_n + g_2(\theta)\bar{V}_n$), and the transforms of the radial accelerations will be $-\xi^2\bar{W}_n$. From (4.22) and (4.23) we see that these expressions are regular in ξ except for a finite number of poles. The theorem of residues can therefore be used to evaluate the integral of (4.9b).

The computation will be carried out in detail for the three lowest modes. The parameters will be taken as

$$\beta = 17, \quad \delta = 0, \quad \lambda^2 = \frac{25}{3}, \quad \nu = .3$$

the values appropriate to a wave of infinite length impinging on a steel shell in water.

Zeroth Mode. For this mode, the motion of the sphere is very simple, consisting of uniform (θ - independent) expansions and contractions of varying amplitude and period.

The stresses, N_φ and N_θ, are given by

$$N_\varphi = N_\theta = \frac{Eh}{1-\nu}\frac{w_o}{R} = \frac{Q_o R}{(1-\nu)\beta} W_o.$$

$$W_o = -\frac{1}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{(1-\nu)k^2\beta e^{i\zeta(\frac{\tau}{\lambda}+1)}d\zeta}{2\zeta(\zeta^3-k^2\zeta-18i\zeta^2+ik^2)}$$

$$\text{Here } k^2 = \frac{2\lambda^2}{1-\nu}; \quad \zeta = \lambda\xi.$$

The integrand is regular except for poles at

$$\zeta = 0, \quad 16.58i, \quad \pm 1 + .71i$$

with residues respectively of

$$1, \quad .0060e^{-16.58(\frac{\tau}{\lambda}+1)}, \quad \text{and} \quad -1.2818e^{-.71(\frac{\tau}{\lambda}+1)}\sin(\tfrac{\tau}{\lambda} + 1.89).$$

Jordan's lemma can be adapted to give

$$W_o = \frac{-(1-\nu)\beta}{2}\left[1.00 - 1.2818e^{-.71(\frac{\tau}{\lambda}+1)}\sin(\tfrac{\tau}{\lambda} + 1.89) - 0.0060e^{-16.58(\frac{\tau}{\lambda}+1)}\right]. \qquad (5.1)$$

A plot of $-2N_\varphi/RQ_o$ appears in Figure 2.[5]

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5. These scales will also be used inplotting the 1st and 2nd modes. The reason for their choice will become clear on page 18.

The radial acceleration is found directly from (5.1) to be

$$\frac{\partial^2 w_o}{\partial \tau^2} = \frac{Q_o R^2}{Eh\beta}\frac{\partial^2 W_o}{\partial \tau_2} = -\frac{Q_o R^2}{4Eh}\left\{e^{-.71(\frac{\tau}{\lambda}+1)}\left[0.0941\sin(\frac{\tau}{\lambda}+1.89)\right.\right.$$
$$\left.\left. + 0.2697\cos(\frac{\tau}{\lambda}+1.89)\right] - 0.2427^{-16.58(\frac{\tau}{\lambda}+1)}\right\}. \quad (5.2)$$

The plot of $-\frac{4Eh}{Q_o R^2}\frac{\partial^2 w}{\partial \tau^2}\Big]_o$ appears in Figure 5.[5]

There is an alternative to the computational scheme we have used, in which the integrand is expanded in a power series about infinity and the series integrated term by term. Since there was a possibility that it might prove simpler, this was tried out, the zeroth mode acceleration being taken as a test case.

$$\frac{\partial^2 W_o}{\partial \tau^2} = \frac{1}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{\beta\zeta e^{i\zeta(\frac{\tau}{\lambda}+1)}d\zeta}{\zeta^3 - k^2\zeta - 18i\zeta^2 + ik^2}$$

$$-\frac{4Ehw_{\tau\tau}}{Q_o R^2}\Big]_o = -\frac{4}{2\pi}\int_{(\infty)} e^{i\zeta(\frac{\tau}{\lambda}+1)}\left[\zeta^{-2} + 18i\zeta^{-3}\right.$$
$$\left. - 299\zeta^{-4} - 49571i\zeta^{-5} + 82201\zeta^{-6} + \ldots\right]d\zeta$$
$$= \left[(\frac{\tau}{\lambda}+1) - 36(\frac{\tau}{\lambda}+1)^2 + \frac{2}{3}(299)(\frac{\tau}{\lambda}+1)^3 + \ldots\right]. \quad (5.3)$$

The results, which appear in Table 1, demonstrate the impracticality of this new procedure. It is apparent that at least 12 terms are needed to find only the first maximum to within 10%

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5 These scales will also be used in plotting the 1st and 2nd modes. The reason for their choice will become clear on page 18.

First Mode

The stress components here are

$$N_\varphi = N_\theta = \frac{Eh}{1-\nu}\left[\frac{w}{R} + \frac{1}{R}\frac{\partial v}{\partial \theta}\right]_1 = \frac{Eh}{(1-\nu)}\left[\frac{w}{R} + \frac{v\cot\theta}{R}\right]_1$$

$$= \frac{Q_o R}{(1-\nu)\beta}\left[W + \frac{\partial V}{\partial \theta}\right]_1 = \frac{Q_o R}{(1-\nu)\beta}\left[W_1 - V_1\right]\cos\theta.$$

$$W_1 - V_1 = -\frac{3\beta}{2\pi i}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{2e^{i\zeta\frac{\tau}{\lambda}}\,d\zeta}{2i\zeta^4 + 38\zeta^3 - 1131i\zeta^2 - 575\zeta + 575i}$$

The integrand has poles at

$$\zeta = 1.17i, \quad 16.58i, \quad \text{and} \quad \pm 3.81 + 0.645i.$$

On computing the residues we obtain

$$W_1 - V_1 = 1.9666e^{-1.17(\frac{\tau}{\lambda}+1)} - 0.1076e^{-16.58(\frac{\tau}{\lambda}+1)}$$

$$-1.8488e^{-0.625(\frac{\tau}{\lambda}+1)}\sin\left[3.811(\tfrac{\tau}{\lambda}+1) + 1.479\right]. \qquad (5.4)$$

Figure 3 shows a plot of $-2N_\varphi/RQ_o$ at the point $\theta = \pi$.

The radial acceleration is found most easily by inverting the transform, $-\xi^2\overline{W}_1\cos\theta$.

$$\left[\frac{\partial^2 w}{\partial \tau_2}\right]_1 = \left[\frac{Q_o R^2}{Eh\beta}\frac{\partial^2 W}{\partial \tau^2}\right]_1 = \frac{Q_o R^2}{Eh\beta}\frac{\partial^2 W_1}{\partial \tau^2}\cos\theta$$

$$\frac{\partial^2 W_1}{\partial \tau^2} = \frac{1}{2\pi i}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{(102\zeta^2 - 1275)e^{i\zeta(\frac{\tau}{\lambda}+1)}\,d\zeta}{2i\zeta^4 + 38\zeta^3 - 1131i\zeta^2 - 575\zeta + 575i}$$

$$= 1.064e^{-0.625(\frac{\tau}{\lambda}+1)}\sin\left[2.72 + 3.8109(\tfrac{\tau}{\lambda}+1)\right]$$

$$-3.5335e^{-16.58(\frac{\tau}{\lambda}+1)} + 3.1171e^{-1.17(\frac{\tau}{\lambda}+1)} \qquad (5.5)$$

$$-\frac{4Eh}{Q_o R^2}\frac{\partial^2 w}{\partial \tau^2}\Big]_1$$ is plotted for $\theta = \pi$ in Figure 6.

Second Mode.

In general, the two tangential stress components are not equal to each other for this mode. We have, in fact,

$$N_\theta = \frac{Eh}{1-\nu^2}\left[\varepsilon_\theta + \nu\varepsilon_\varphi\right], \quad N_\varphi = \frac{Eh}{1-\nu^2}\left[\varepsilon_\varphi + \omega\varepsilon_\theta\right],$$

where

$$\varepsilon_\theta = \frac{1}{R}\left[w + \frac{\partial v}{\partial\theta}\right]_2 = \frac{Q_o R}{Eh\beta}\left[W + \frac{\partial V}{\partial\theta}\right]_2 = \frac{Q_o R}{Eh\beta}\left[3\cos^2\theta(-2V_2 + \tfrac{1}{2}W_2) + (3V_2 - \tfrac{1}{2}W_2)\right]$$

$$\varepsilon_\varphi = \frac{1}{R}\left[w + v\cot\theta\right]_2 = \frac{Q_o R}{Eh\beta}\left[W + V\cot\theta\right]_2$$

$$= \frac{Q_o R}{Eh\beta}\left[3\cos^2\theta(-V_2 + \tfrac{1}{2}W_2) - \tfrac{1}{2}W_2\right].$$

$$W_2 = \frac{1}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{[676.8125\zeta^3 - 34491.40625\zeta]e^{i\zeta(\frac{\tau}{\lambda}+1)}d\zeta}{D}$$

$$V_2 = -\frac{1}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty} \frac{8460.15625\zeta e^{i\zeta(\frac{\tau}{\lambda}+1)}\,d\zeta}{D}$$

where

$$D = -.91\zeta^7 + 19.11\, i\zeta^6 + 123.725\zeta^5 - 1119.475\, i\zeta^4 - 3292.5\zeta^3 + 4212.25\, i\zeta^2 + 2756.25\zeta - 2756.25i \qquad (5.6)$$

The denominators have seven zeros:

$$16.54\, i, \quad \pm 7.35 + 0.43\, i, \quad \pm 0.86 + 1.79\, i, \quad \pm 0.92 + 0.007\, i.$$

The theorem of residues gives

$$W_2 = 0.213e^{-16.54(\frac{\tau}{\lambda}+1)} - 2e^{-0.43(\frac{\tau}{\lambda}+1)}\left\{0.0426\left[\sin 7.3483(\tfrac{\tau}{\lambda}+1)\right] - 0.0254\cos\left[7.3483(\tfrac{\tau}{\lambda}+1)\right]\right\} + 2e^{-1.79(\frac{\tau}{\lambda}+1)}\left\{12.2253\sin\left[0.8578(\tfrac{\tau}{\lambda}+1)\right] + 2.8318\cos\left[0.8579(\tfrac{\tau}{\lambda}+1)\right]\right\} - 2e^{-.007(\frac{\tau}{\lambda}+1)}\left\{3.6145\sin\left[0.9242(\tfrac{\tau}{\lambda}+1)\right] + 2.9634\cos\left[0.9242(\tfrac{\tau}{\lambda}+1)\right]\right\} \quad (5.7)$$

$$V_2 = 0.0082e^{-16.54(\frac{\tau}{\lambda}+1)} - 2e^{0.43(\frac{\tau}{\lambda}+1)}\left\{0.0102\sin\left[7.35(\tfrac{\tau}{\lambda}+1)\right] + 0.0888\cos\left[7.35(\tfrac{\tau}{\lambda}+1)\right]\right\} + 2e^{-1.79(\frac{\tau}{\lambda}+1)}\left\{2.8117\left[\sin 0.8578(\tfrac{\tau}{\lambda}+1)\right] + 0.8242\cos\left[0.8579(\tfrac{\tau}{\lambda}+1)\right]\right\} - 2e^{-.007(\frac{\tau}{\lambda}+1)}\left\{0.9015\cdot\sin\left[0.9242(\tfrac{\tau}{\lambda}+1)\right] + 0.7395\cos\left[0.9242(\tfrac{\tau}{\lambda}+1)\right]\right\}. \quad (5.8)$$

At $\Theta = \pi$ where the greatest stresses occur, $N_\Theta = N_\varphi = \frac{Q_oR}{(1-\nu)\beta}(W_2-3V_2)$. A plot of $-\frac{2N_\varphi}{Q_oR}$ for $\Theta = \pi$ appears in Figure 4.

The radial acceleration is given by

$$\left.\frac{\partial^2 w}{\partial\tau^2}\right]_2 = \frac{Q_oR^2}{Eh\beta}\left.\frac{\partial^2 W}{\partial\tau^2}\right]_2 = \frac{Q_oR^2}{Eh\beta}\frac{\partial^2 W_2}{\partial\tau^2}\left(\frac{3\cos 2\Theta+1}{4}\right)$$

$$\frac{\partial^2 W_2}{\partial\tau^2} = -\frac{1}{2\pi\lambda^2}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty}\frac{[676.8125\zeta^5-34491.40625\zeta^3]e^{i\zeta(\frac{\tau}{\lambda}+1)}d\zeta}{D}$$

where D is given in (5.6).

$$\frac{\partial^2 W_2}{\partial\tau^2} = 6.6576\,e^{-16.54(\frac{\tau}{\lambda}+1)} + e^{-0.43(\frac{\tau}{\lambda}+1)}\left\{0.564\sin\left[7.3483(\tfrac{\tau}{\lambda}+1)\right] - 0.2457\cos\left[7.3483(\tfrac{\tau}{\lambda}+1)\right]\right\} + e^{-1.79(\frac{\tau}{\lambda}+1)}\left\{8.939\sin\left[0.8578\cdot(\tfrac{\tau}{\lambda}+1)\right] - 6.9956\cos\left[0.8578(\tfrac{\tau}{\lambda}+1)\right]\right\} + e^{-0.007(\frac{\tau}{\lambda}+1)}\left\{0.6982\sin\left[0.9242(\tfrac{\tau}{\lambda}+1)\right] + 0.5966\cos\left[0.9242(\tfrac{\tau}{\lambda}+1)\right]\right\}. \quad (5.9)$$

$-\frac{4Eh}{Q_o R^2}\frac{\partial^2 w}{\partial \tau^2}\Big]_2$ is plotted for $\Theta = \pi$ in Figure 7.

If we examine Figures 2, 3, and 4 it becomes evident that the stresses in the lowest mode are very much greater than those of the 1st and 2nd modes, and very likely those of the higher modes as well. In this connection it should be recalled that the plots in Figures 3 and 4, which were made for $\Theta = \pi$, show the largest stresses which can occur in the first and second modes at any time. We may therefore consider the resultant stresses in the sphere to be predominantly those of the zeroth mode.

The case of the acceleration is not so simple. A picture of the total acceleration cannot be gained by looking at the lower modes. In fact, it would seem from Figures 5, 6, and 7, that the series, $\sum_{n=0}^{\infty} \frac{\partial^2 w_n}{\partial \tau^2}$ does not represent the total acceleration for all τ. The rate of change of radial acceleration is very great for the first three modes at $\tau/\lambda = -1$, $\Theta = \pi$; probably the total radial acceleration as summed mode by mode will be discontinuous at $\tau = -\lambda$, $\Theta = \pi$. This indicates that $\sum_{n=0}^{\infty} \frac{\partial^2 w_n}{\partial \tau^2}$ does not converge uniformly, and therefore that $\frac{\partial^2 w}{\partial \tau^2} \neq \sum_{n=0}^{\infty} \frac{\partial^2 w_n}{\partial \tau^2}$ This difficulty, associated with the use of the series expansion as a method of solution, will be encountered again in Section VI when we discuss the resultant pressure distribution. By anticipating the results of that section, we can find the initial value of $\frac{\partial^2 w}{\partial \tau^2}$ at $\Theta = \pi$.

Equation (2.2) gives, for $\tau/\lambda = -1$

$$S = \rho h \frac{\partial^2 w}{\partial \tau^2} = \frac{\rho h}{R^2 \rho} \frac{\partial^2 w}{\partial \tau^2}$$

$$\frac{-4S}{Q_o} = - \frac{4Eh}{Q_o R^2} \frac{\partial^2 w}{\partial \tau^2} .$$

Taking

$$S = -2Q_o \qquad ((6.9),\ \theta = \pi)$$

we have

$$- \frac{4Eh}{Q_o R^2} \frac{\partial^2 w}{\partial \tau^2} = 8.$$

Thus, while we can obtain a very good approximation of the stresses by considering only the lowest mode, the same is definitely not true of the acceleration. The actual initial acceleration is about 40 times as great as the maximum acceleration in the zeroth mode. The agreement obtained by considering the first and second modes along with the zeroth is not appreciably better; the results still differ by more than a factor of 6.

Quasi-Static Case.

It is of interest to compare the hoop stresses we have obtained, as represented by those of the zeroth mode, with the stresses for the quasi-static case; i.e. for the case in which the sphere is taken to be rigid and scattering is neglected.

Since the effects of only the incident wave are considered, we have for the pressure at time t

$$\text{pressure} = \begin{cases} \dfrac{Q_o 2\pi R(tc+R)}{4\pi R^2} = \dfrac{Q_o(\frac{\tau}{\lambda}+1)}{2} & -1 \le \frac{\tau}{\lambda} \le 1. \\ Q_o & +1 \le \frac{\tau}{\lambda} < \infty . \end{cases}$$

The stresses are given by

$$N_\theta = N_\varphi = \begin{cases} -\dfrac{RQ_o(\frac{\tau}{\lambda}+1)}{4} & -1 \le \dfrac{\tau}{\lambda} + 1 \\ -\dfrac{RQ_o}{2} & +1 < \dfrac{\tau}{\lambda} < \infty \end{cases}$$

$-\dfrac{2N_\varphi}{RQ_o}$ appears as the dotted line in Figure 2, where it can be seen that the very simple quasi-static case approximates the more exact dynamic case very closely. The stress developments, while slightly out of phase, are essentially parallel with a difference in maxima of only 1%.

For $-1 \le \tau/\lambda \le 1$, there will be an unbalanced force, due to the incident wave, acting on the sphere. This will result in an acceleration of the rigid body in the z-direction. The magnitude of the force is

$$\int_{\theta_o}^{\pi}\int_0^{2\pi} R^2 Q_o \cos\theta \sin\theta \, d\varphi = Q_o \pi R^2 \sin^2\theta_o$$

$$= Q_o \pi [R^2 - (tc)^2]$$

$$= Q_o \pi R^2 [1 - (\tau/\lambda)^2].$$

Therefore

$$4\pi R^2 h\rho \frac{\partial^2 z}{\partial \tau^2} = Q_o \pi R^2 [1 - (\tfrac{\tau}{\lambda})^2]$$

$$\frac{4Eh}{R^2}\frac{\partial^2 z}{\partial \tau^2} = Q_o [1 - (\tfrac{\tau}{\lambda})^2]$$

$$\frac{4Eh}{Q_o R^2}\frac{\partial^2 z}{\partial \tau^2} = [1 - (\tfrac{\tau}{\lambda})^2].$$

This has been plotted for purposes of comparison in Figures 5, 6, and 7 (dotted curves).

The maximum acceleration in the zeroth mode is only one fifth that of the rigid sphere. The resemblance is greater for the first mode, where the two significant maxima occur within the time $|\frac{\tau}{\lambda}|<1$, the larger being one half that of the rigid sphere. In the second mode where we begin to have large negative accelerations, there is considerable difference in form between the dynamic and quasi-static cases although the maximum positive acceleration of the former, .7, has moved still closer to the rigid body value of 1.

This value of 1 is, we recall, very much smaller than the maximum of the <u>total</u> radial acceleration for the dynamic case (p. 17).

<u>VI. Resultant pressure distribution.</u>

From (3.2), (4.2), (4.15), (4.24) and (4.9b) the total pressure is known to be:

$$P_{total} = P_I(\text{Incident}) + P_{II} = -p^*\varphi_\tau = -Q_o X_\tau$$

$$= Q_o - \frac{iQ_o}{2\pi}\int_{-i\alpha-\infty}^{-i\alpha+\infty} e^{i\xi\tau}\,\bar{\psi}\xi\,d\xi$$

$$= Q_o + \frac{iQ_o}{2\pi}\int_{-i\alpha+\infty}^{-i\alpha+\infty} \xi e^{i\xi\tau}\sum\frac{(-i)^n(2n+1)}{2\xi^2}\,h_n^{(2)}(\lambda\xi r)\left[1 + \right.$$

$$\left. + \frac{C_1\lambda h_n^{(1)'} + iC_2 h_n^{(1)}}{C_1\lambda h_n^{(2)'} + C_2 h_n^{(2)}}\right] P_n(\cos\Theta)\,d\xi \qquad (6.1)$$

where C_1 and C_2 are given on p. 10.

Unfortunately, the order of summation and integration in (6.1) cannot be interchanged, i.e. the total pressure cannot be found as the sum of the pressures associated with the

individual vibrational modes. This can be seen for the specific case $r = 1$, $\tau = -\lambda$, and $\Theta = \pi$ as follows. At the moment of impact, $\tau = -\lambda$, we have for each mode and for all Θ, $v_n = w_n = \frac{\partial^2 v_n}{\partial t^2} = \frac{\partial^2 w_n}{\partial t^2} = 0$. Our equations of equilibrium, (2.9) and (2.10) require that the pressure at the surface of the sphere must likewise vanish for each mode at $\tau = -\lambda$, so that the total pressure would be zero for all Θ. We should expect however, from what is known of the theory of scattering of plane waves, that the pressure Q_o would be doubled for $\Theta = \pi$ and not reduced to zero.

Alternatively, we recall from p. 16 that the total radial acceleration is discontinuous at $\tau = -\lambda$, $\Theta = \pi$. Therefore the total pressure as summed mode by mode will be discontinuous, indicating the non-uniformity of convergence of the series in (6.1).

Because of this peculiarity in convergence, it is not possible to obtain an approximate solution for the pressure by considering just the first few terms of the series. $\bar{\psi}$, (4.15), must be found in closed form if P_{total} is to be evaluated. This has not as yet proved feasible because of the complexity of the summation which must be made. It was noted however that the expansions for $\bar{\psi}^o$ and $\bar{\psi}$ are very similar for $r = 1$, and ξ very large or $(\frac{\tau}{\lambda} - \cos\Theta)$ very small. This fact can be used to obtain the pressure at the surface of the sphere for $\tau/\lambda \simeq \cos\Theta$.

By comparison with the expression for $\bar{\psi}_o$ on page 9 it is seen that the pressure associated with the incident wave may be written

$$P_I = Q_o = -\frac{Q_{oi}}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty}\frac{e^{i\zeta\tau/\lambda}}{\zeta}\sum_{n=0}^{\infty}\delta_{jn}(\lambda\xi r)P_n(\cos\theta)(2n+1)(-i)^n\,d\zeta\,.$$

For $\zeta >> \beta$ and $r = 1$, P_{II} (cf. 6.1) becomes, to first order

$$P_{II} = \frac{Q_o}{2\pi}\int_{-i-\infty}^{-i\alpha\lambda+\infty}\frac{e^{i\zeta\tau/\lambda}}{\zeta}\sum_{n=0}^{\infty}\frac{\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta}(2n+1)(-i)^nP_n(\cos\theta)d\zeta \tag{6.2}$$

and P_I reduces to

$$P_I = -\frac{Q_{oi}}{2\pi}\int_{-i\alpha\lambda-\infty}^{-i\alpha\lambda+\infty}\frac{e^{i\zeta\tau/\lambda}}{\zeta}\sum_{n=0}^{\infty}\frac{\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta}(2n+1)(-i)^nP_n(\cos\theta)d\zeta. \tag{6.3}$$

Equations (6.4) and (4.14) tell us that

$$\zeta e^{-i\zeta\cos\theta} \xrightarrow[\zeta\to\infty]{} \sum_{n=0}^{\infty}\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})(2n+1)(-i)^nP_n(\cos\theta).$$

Therefore, take

$$(\zeta - \tfrac{\pi}{2})e^{-i(\zeta-\pi/2)\cos\theta} \xrightarrow[\zeta\to\infty]{} \sum_{n=0}^{\infty}\sin(\zeta - \tfrac{n\pi}{2} - \tfrac{\pi}{2})(2n+1)\cdot(-i)^nP_n(\cos\theta)$$

so that to first order:

$$P_{II} = \frac{Q_o}{2\pi}\int\frac{e^{i\zeta(\frac{\tau}{\lambda} - \cos\theta)}}{\zeta}e^{i\frac{\pi}{2}\cos\theta}\,d\zeta$$

$$= Q_oe^{i\frac{\pi}{2}(\cos\theta + 1)} \qquad \text{for} \quad 0 \le (\tfrac{\tau}{\lambda} - \cos\theta) << .05$$

$$= 0 \qquad \text{for} \quad (\tfrac{\tau}{\lambda} - \cos\theta) < 0 \tag{6.4}$$

Higher order terms may be obtained in the same way.

To second order:

$$P_{II} = \frac{Q_o}{2\pi}\int\frac{e^{i\zeta\tau/\lambda}}{\zeta}\sum_{n=0}^{\infty}\left[(\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})(1 - \frac{in(n+1)}{2\zeta^2}) + \frac{e^{i(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}[2 + 2\beta + n(n+1)]}{2\zeta^2}\right](2n+1)(-i)^nP_n\,d\zeta$$

$$= \frac{Q_o}{2\pi}\int \frac{e^{i\zeta\tau/\lambda}}{\zeta}\sum_{n=0}^{\infty}\left[\frac{\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta} + \frac{\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})n(n+1)}{2\zeta^2}\right.$$

$$\left. + \frac{\beta+1}{\zeta^2} e^{i(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}\right](2n+1)(-i)^n P_n(\cos\theta)d\zeta \qquad (6.5)$$

$$P_I = -\frac{Q_{oi}}{2\pi}\int \frac{e^{i\zeta\tau/\lambda}}{\zeta}\sum_{n=0}^{\infty}\left[\frac{\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta} - \frac{\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})n(n+1)}{2\zeta^2}\right]$$

$$\cdot(2n+1)(-i)P_n(\cos\theta)d\zeta\,. \qquad (6.6)$$

In this case

$$e^{-i\zeta\cos\theta} \xrightarrow[\zeta\to\infty]{} \sum_{n=0}^{\infty}(2n+1)(-i)^n P_n(\cos\theta)\left[\frac{\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta}\right.$$

$$\left. - \frac{\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})n(n+1)}{2\zeta^2}\right]$$

therefore by analogy, to second order:

$$e^{-i(\zeta - \frac{\pi}{2})\cos\theta} \xrightarrow[\zeta\to\infty]{} \sum_{n=0}^{\infty}(2n+1)(-i)^n P_n\left[\frac{\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta}[1 + \frac{\pi}{2\zeta}]\right.$$

$$\left. + \frac{\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})n(n+1)}{2\zeta^2}\right].$$

This does not correspond exactly to the first two terms in P_{II}, therefore we must subtract

$$\frac{\frac{\pi}{2}\, e^{-i(\zeta - \frac{\pi}{2})\cos\theta}}{\zeta} \xrightarrow[\zeta\to\infty]{} \sum_{n=0}^{\infty}\frac{\pi}{2\zeta^2}\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})(2n+1)(-i)^n P_n(\cos\theta).$$

We also have, again to second order

$$(\beta+1)\left[\frac{e^{-i\zeta\cos\theta}}{\zeta} + \frac{ie^{-i(\zeta - \frac{\pi}{2})\cos\theta}}{\zeta}\right] \xrightarrow[\zeta\to\infty]{} (\beta+1)\sum_{n=0}^{\infty}\left[\frac{\cos(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta^2}\right.$$

$$\left. + \frac{i\sin(\zeta - \frac{n\pi}{2} - \frac{\pi}{2})}{\zeta^2}\right](2n+1)(-i)^n P_n(\cos\theta).$$

P_{II} can now be written:

$$P_{II} = \frac{Q_o}{2\pi}\int \frac{e^{i\zeta\tau/\lambda}}{\zeta}\left[e^{-i(\zeta-\pi/2)\cos\theta} - \frac{\pi}{2}\frac{e^{-i(\zeta-\frac{\pi}{2})\cos\theta}}{\zeta} + (\beta+1)\left(\frac{e^{-i\zeta\cos\theta}}{\zeta} + \frac{ie^{-i(\zeta-\frac{\pi}{2})\cos\theta}}{\zeta}\right)\right]d\zeta$$

$$P_{II} = \frac{Q_o}{2\pi}\int \frac{e^{i\zeta(\frac{\tau}{\lambda}-\cos\theta)}}{\zeta}\left[e^{i\frac{\pi}{2}\cos\theta} + \frac{\beta+1}{\zeta}(1 + ie^{i\frac{\pi}{2}\cos\theta}) - \frac{\pi}{2\zeta}e^{i\frac{\pi}{2}\cos\theta}\right]$$

$$= Q_o\left[e^{i\frac{\pi}{2}(\cos\theta+1)} - (\tfrac{\tau}{\lambda}-\cos\theta)(1+\beta)(1 + e^{i\frac{\pi}{2}(\cos\theta+1)} + \frac{\pi}{2}(\tfrac{\tau}{\lambda}-\cos\theta)e^{i\frac{\pi}{2}\cos\theta}\right] \quad \text{for} \quad 0 \le (\tfrac{\tau}{\lambda}-\cos\theta) << .05$$

$$= 0 \qquad \text{for} \quad (\tfrac{\tau}{\lambda}-\cos\theta) < 0. \qquad (6.7)$$

Additional terms will be of little value since our expansion is valid only for $\zeta >> \beta$, or $(\tau/\lambda - \cos\theta) << .05$. The terms we have found so far however are sufficient to tell us some things of importance.

The incoming wave will reach the point $(1,\theta)$ on the sphere at time $\tau/\lambda = \cos\theta$. The initial pressure for each θ is given by[6]

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6. The elastic waves in the shell will travel more rapidly than the acoustic waves and will result in a pressure, $P \neq 0$, at $(1,\theta)$ before the time $\tau/\lambda = \cos\theta$, $\theta \neq \pi$. However, this effect is negligibly small compared with the one we are considering.

$$P_{total} = P_I + \mathrm{Re}P_{II}$$

$$= Q_o + Q_o\left\{\cos\left(\frac{\pi}{2}(\cos\theta+1)\right) - 18\left[\frac{\tau}{\lambda} - \cos\theta\right]\left[1+\cos\left(\frac{\pi}{2}(\cos\theta+1)\right)\right]\right.$$

$$\left. + \frac{\pi}{2}\left[\frac{\tau}{\lambda} - \cos\theta\right]\left[\cos\left(\frac{\pi}{2}\cos\theta\right)\right]\right\}. \qquad (6.8)$$

$$= Q_o + Q_o \cos\left[\frac{\pi}{2}(\cos\theta+1)\right]. \qquad (6.9)$$

At $\theta = \pi$, the outermost point of the sphere, the first effect is that of a plane wave hitting a rigid wall so that we have

$$P_{II} = P_I; \quad P_{total} = 2Q_o.$$

At $\theta = \pi/2$, the wave just grazes the sphere and therefore

$$P_{II} = 0; \quad P_{total} = Q_o.$$

As θ varies from π to $\pi/2$, the initial pressure varies continuously from $2Q_o$ to Q_o.

If the sphere were rigid, the steady state pressure distribution would be given by

$$P_{total} = P_I \qquad 0 \leq \theta \leq \pi \ . \qquad [5]$$

The results of section V indicate that we will have asymptotic values of $RQ_o/2$ for the stresses and zero for the radial acceleration, which also correspond to a uniform pressure of $Q_o = P_I$.

At present, this is about all that can be said on the subject of the pressure distribution. A completely satisfactory way of dealing with the problem will not be had until it is possible to find the sum in (6.1).

VII. Conclusions.

Both the stress and the acceleration for each mode can readily be computed to any desired accuracy. For the stress, the zeroth mode is by for the most important and this is closely approximated by the zeroth mode of a simple quasi-static system (p. 18). The acceleration on impact of the outermost portion of the shell ($\theta = \pi$) can be found exactly and is seen to differ markedly from the accelerations associated with the individual vibrational modes.

The problem here considered, apart from its intrinsic interest, should serve as a valuable guide in the solution of similar problems involving obstacles of more complicated shape.

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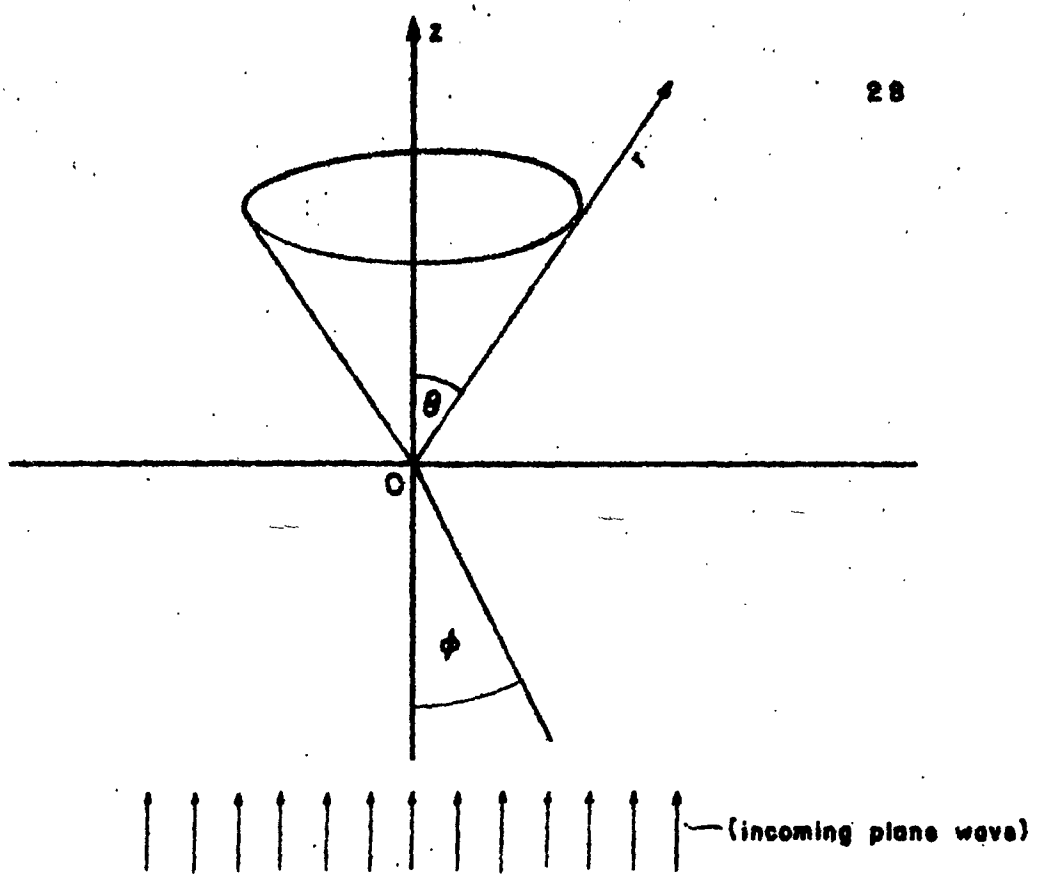
Results to be Approximated

$\frac{\tau}{\lambda} + 1$	$\frac{-4Eh}{Q_o R^2} \frac{\partial^2 w_o}{\partial \tau^2}$
0	0
.1	0.176
.2	0.180
.3	0.155

Results Obtained by Power Series Expansion

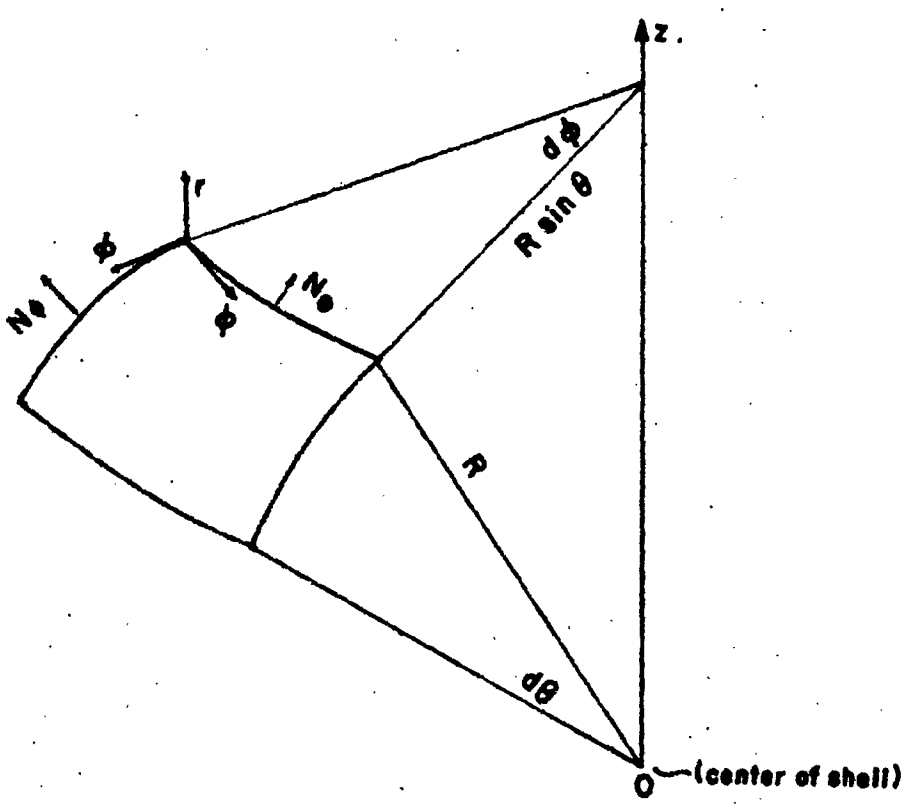
Terms used / $\frac{\tau}{\lambda} + 1$	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0
.1	0.4	0.04	0.239	0.157	0.184	0.177	0.178	0.178	0.178	0.178	0.178	0.178
.2	0.8	-0.64	0.955	-0.367	0.510	0.025	0.255	0.160	0.195	0.183	0.187	0.186
.3	1.2	-2.04	3.342	-3.350	3.308	-2.213	1.711	-0.729	0.620	-0.051	0.252	0.126

Table 1. Approximating the zeroth mode acceleration by a power series expansion.



(a)

Fig. 1. Geometry of the problem.



(b)

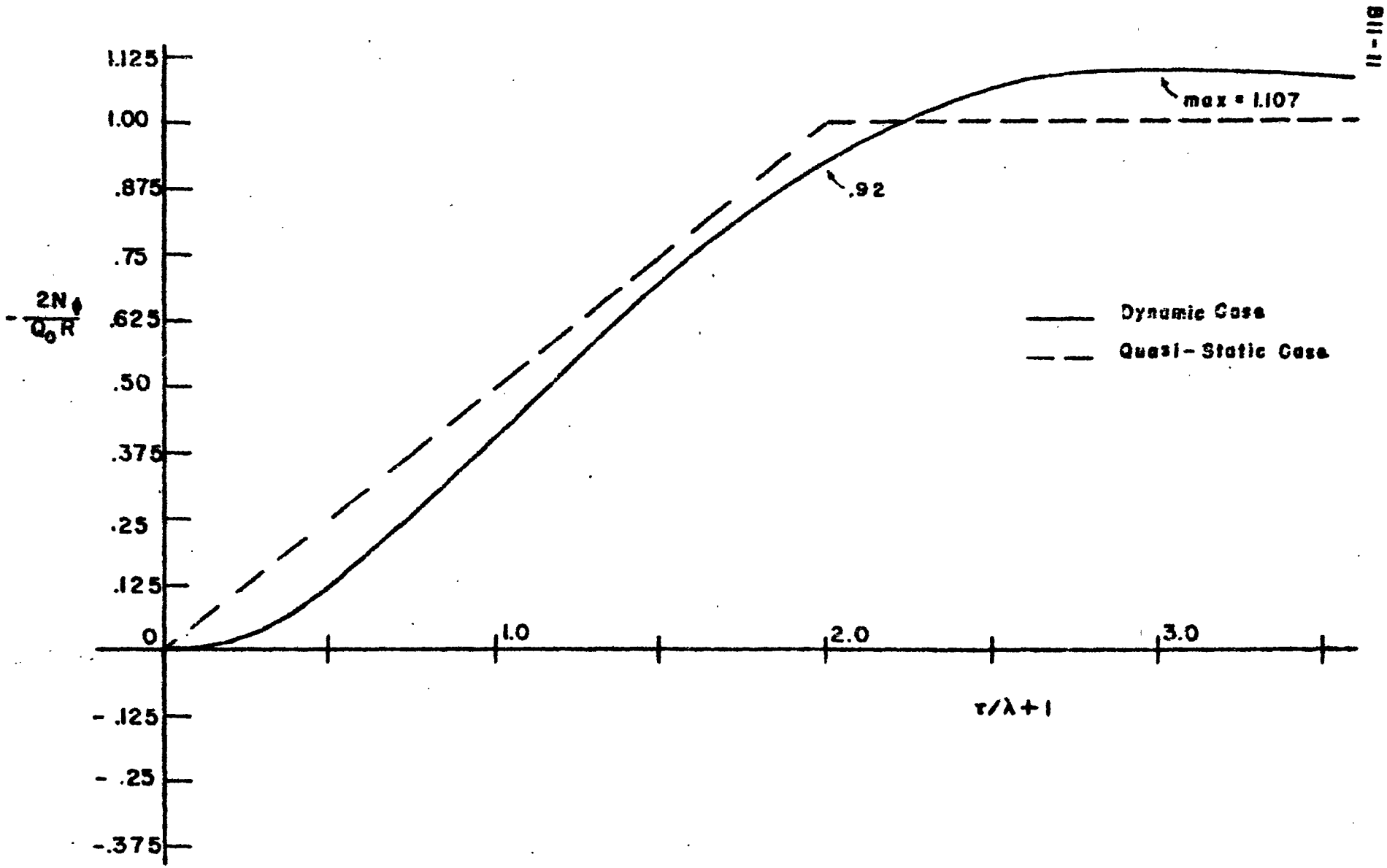


Fig. 2. Hoop stress for the zeroth mode.

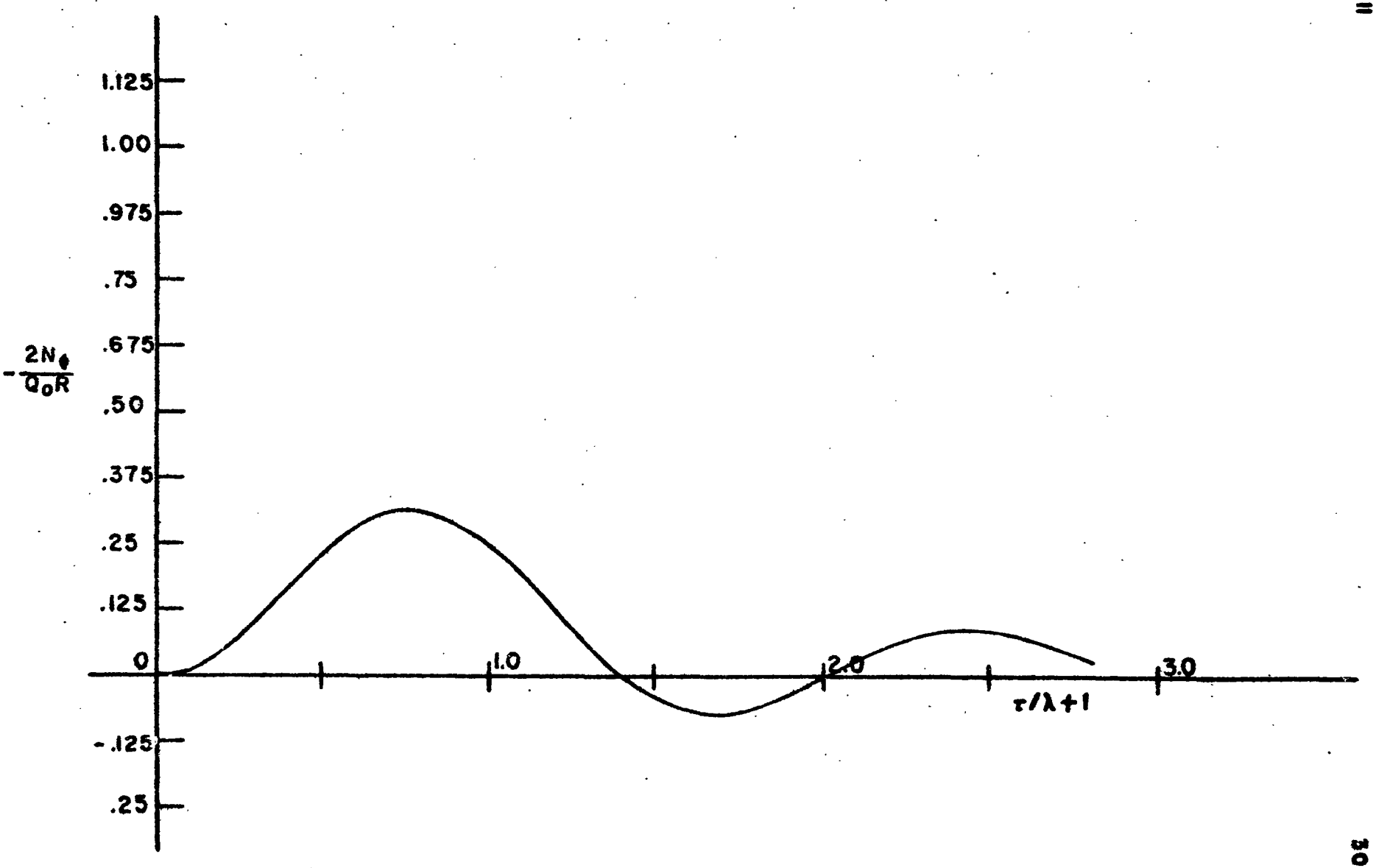


Fig. 3. Hoop stress for the first mode at $\theta = \pi$

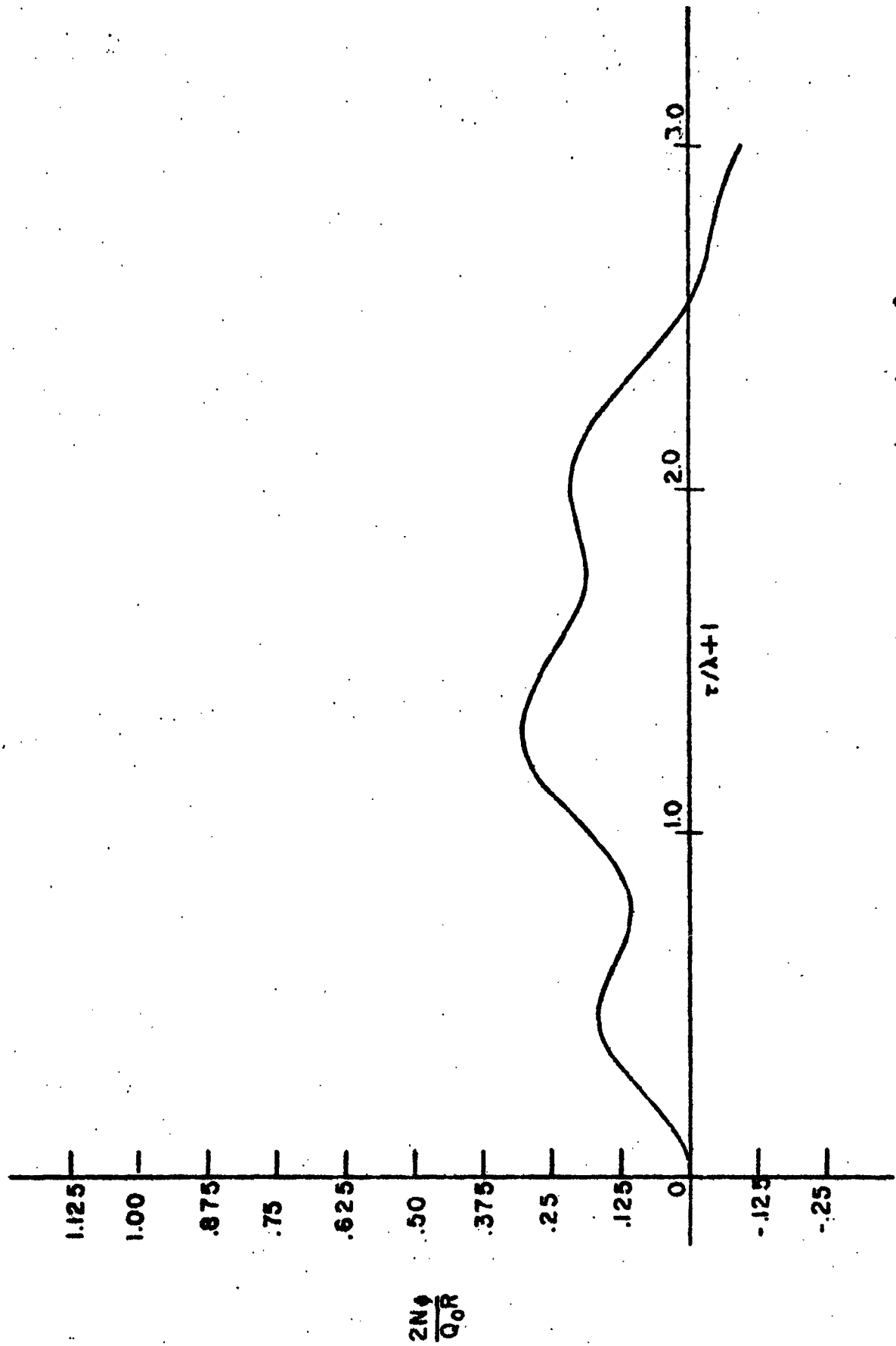


Fig. 4. Hoop stress for the second mode at $\theta = \pi$

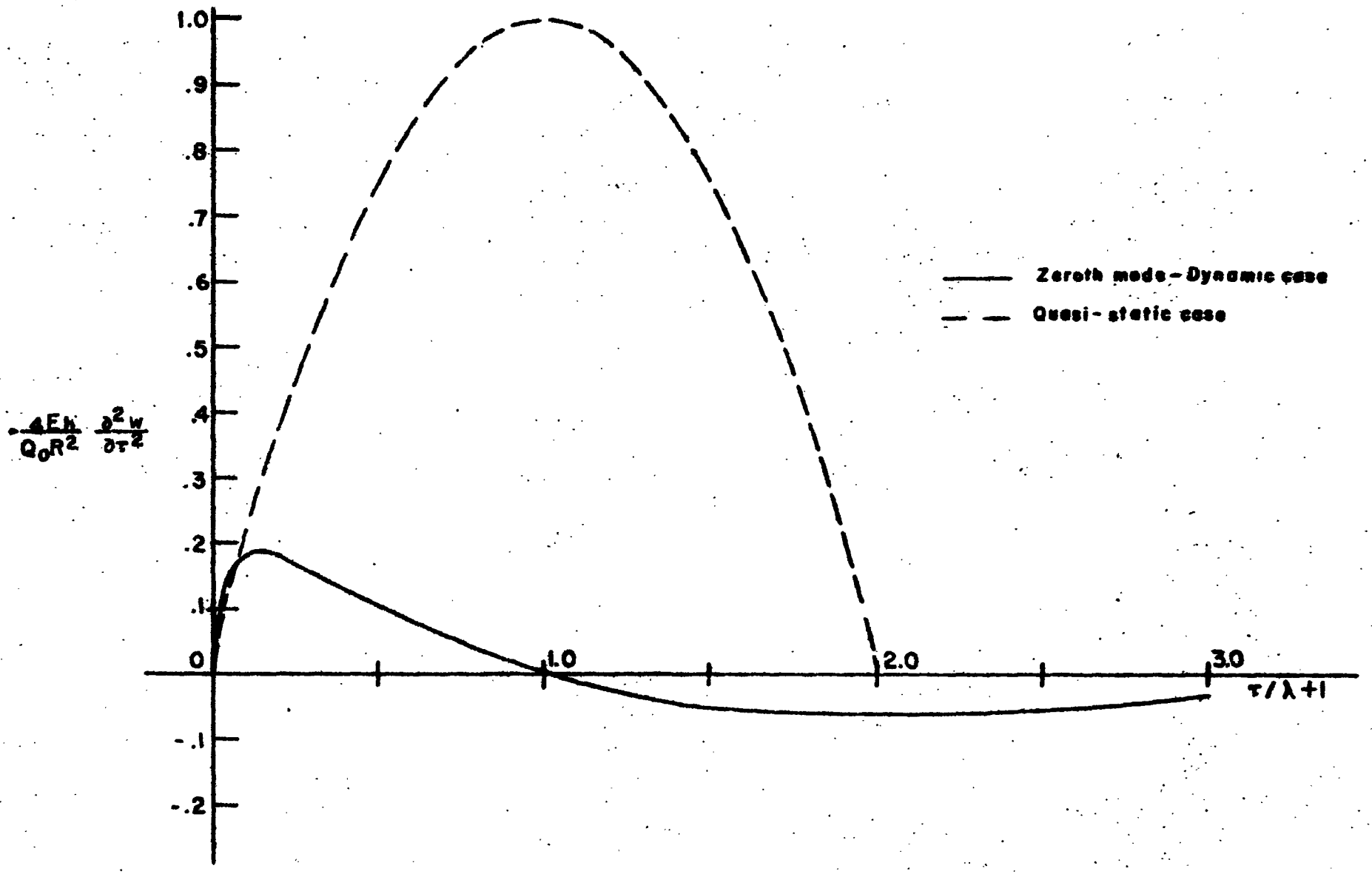


Fig. 5. Radial acceleration for the zeroth mode.

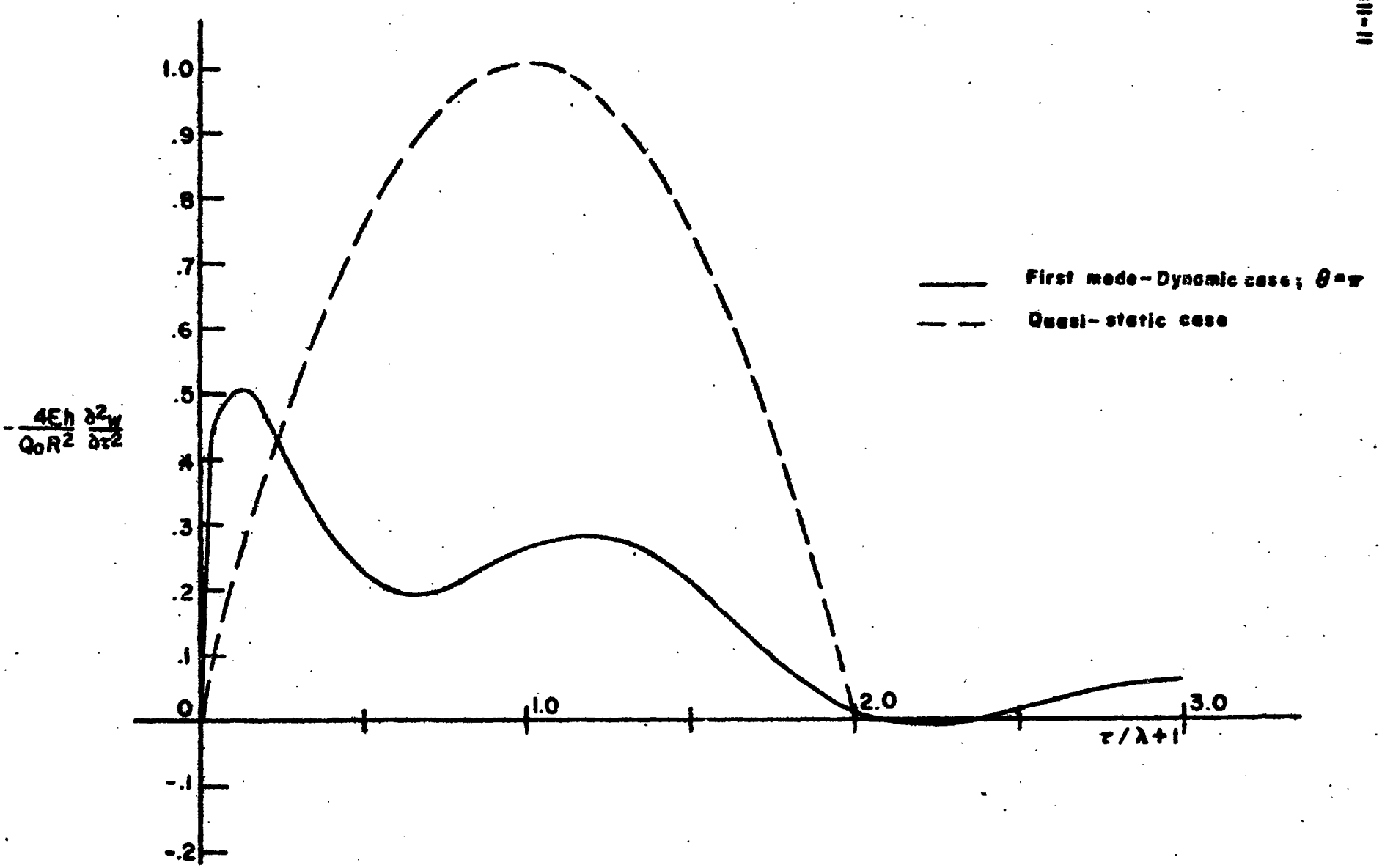


Fig. 6. Radial acceleration for the first mode at $\theta = \pi$

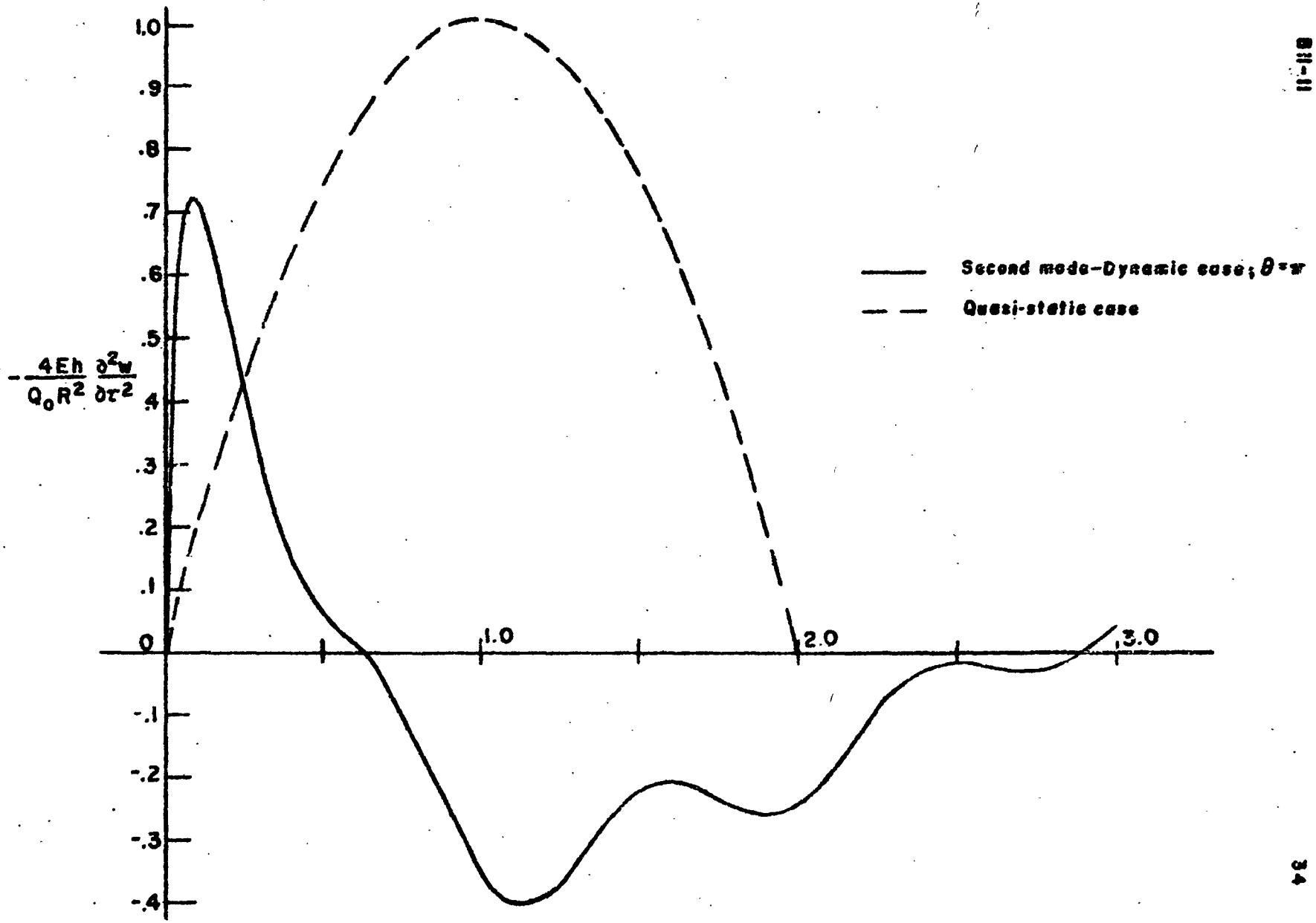


Fig. 7. Radial acceleration for the second mode at $\theta = \pi$

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